Journal of Statistical Physics, Vol. 64, Nos. 3/4, 1991

New Type of Heteroclinic Tangency in Two-Dimensional Maps

Yoshihiro Yamaguchi[1] **and Kiyotaka Tanikawa**[2]

Received May 2, 1989; final March 25, 1991

A new mechanism of heteroclinic tangency is investigated by using two-dimensional maps. First, it is numerically shown that the unstable manifold from a hyperbolic fixed point accumulates to the stable manifold of a nearby period-2 hyperbolic point in a piecewise linear map and that the unstable manifold from a hyperbolic fixed point accumulates to the accumulation of the stable manifold of a nearby period-2 hyperbolic point in a cubic map. Second, a theorem on the impossibility of heteroclinic tangency (in the usual sense) is given for a particular type of map. The notions of *direct* and *asymptotic* heteroclinic tangencies are introduced and heteroclinic tangency is classified into four types.

KEY WORDS: Direct heteroclinic tangency; asymptotic heteroclinic tangency; stable and unstable manifolds; hyperbolic fixed point; basin.

1. INTRODUCTION

In recent years fascinating developments have been made in nonlinear dynamical systems.[1] In this paper, we study the mechanism of the heteroclinic tangency in two-dimensional maps. The heteroclinic tangency as well as the homoclinic tangency in dissipative dynamical systems has been investigated by many authors.[2–5] The heteroclinic tangency (intersection) means that the stable and unstable manifolds from distinct hyperbolic points touch (intersect). However, the detailed structure of the heteroclinic tangency has not been clearly understood compared with the mechanism of the homoclinic tangency.[6–8] Using the results of numerical calculations, we discuss the heteroclinic tangency in a two-dimensional piecewise linear

[1] Liberal Arts and Sciences, Teikyo University of Technology, Ichihara, Chiba 290-01, Japan.
[2] Division of Theoretical Astrophysics, National Astronomical Observatory, Mitaka, Tokyo 181, Japan.

0022-4715/91/0800-0741$06.50/0

map similar to the Lozi map[9] in Section 2, and study the tangency for a two-dimensional cubic map in Section 3. The notions of direct and asymptotic heteroclinic tangencies are introduced in Section 4. We state a theorem on the existence of heteroclinic tangency and give its proof. The theorem implies the existence of asymptotic heteroclinic tangency. We show that the heteroclinic tangency is classified into four types. In the last section, the effect of the heteroclinic tangency on the orbit and the structure of basins is discussed.

2. THE HETEROCLINIC TANGENCY IN A TWO-DIMENSIONAL PIECEWISE LINEAR MAP

When we investigate the heteroclinic tangency between the stable and unstable manifolds by numerical calculations, it is very convenient to use the maps in which the coexistence of the hyperbolic saddle points is clearly visible. Thus we take the piecewise linear map and the two-dimensional cubic map instead of the famous Hénon map.[10]

In this section, we study the mechanism of the heteroclinic tangency for the piecewise linear map $T_a(X, Y)$:

$$T_a\colon\ X_{n+1} = Y_n, \quad Y_{n+1} = f_a(Y_n) - JX_n \tag{1}$$

$$f_a(Y) = \begin{cases} -aY + a & \text{for} \quad Y \geqslant 1/2 \\ aY & \text{for} \quad -1/2 \leqslant Y < 1/2 \\ -aY - a & \text{for} \quad Y \leqslant -1/2 \end{cases}$$

where J $(0 < J \leqslant 1)$ is the Jacobian determinant and a (>0) is a bifurcation parameter. We consider the parameter range $a > J + 1$. Then, the origin $(0, 0)$ is a hyperbolic fixed point without reflection. This map has the period-2 hyperbolic points without reflection $S_1(X^*, Y^*)$ and $S_2(-X^*, -Y^*)$, where X^* and Y^* are given by

$$Y^* = -X^* = a/(a - J - 1) \tag{2}$$

For the piecewise linear map, it is easy to calculate the exact expressions of the unstable and stable manifolds in the vicinity of hyperbolic points. Using linear stability analysis, we can obtain the expressions of the unstable manifold $\mathbf{W}_\mathrm{u}^1$ and stable manifold $\mathbf{W}_\mathrm{s}^1$ around the origin:

$$Y = \alpha X \qquad \text{for} \quad \mathbf{W}_\mathrm{u}^1(|X| \leqslant 1/2) \tag{3}$$

$$Y = (J/\alpha)\, X \qquad \text{for} \quad \mathbf{W}_\mathrm{s}^1(|Y| \leqslant 1/2) \tag{4}$$

where $\alpha = [a + (a^2 - 4J)^{1/2}]/2$.

The expressions of $\mathbf{W}_u^2$ and $\mathbf{W}_s^2$ are given by

$$Y = -\alpha(X + X^*) - Y^* \quad \text{for } \mathbf{W}_u^2 \text{ from } S_2\ (X \geqslant 1/2) \tag{5}$$

$$Y = -\alpha(X - X^*) + Y^* \quad \text{for } \mathbf{W}_u^2 \text{ from } S_1\ (X \leqslant -1/2) \tag{6}$$

$$Y = (-J/\alpha)(X + X^*) - Y^* \quad \text{for } \mathbf{W}_s^2 \text{ from } S_2\ (Y \leqslant -1/2) \tag{7}$$

$$Y = (-J/\alpha)(X - X^*) + Y^* \quad \text{for } \mathbf{W}_s^2 \text{ from } S_1\ (Y \geqslant 1/2) \tag{8}$$

Using Eqs. (3)–(8) and the map (1), we can formally construct $\mathbf{W}_u^1$, $\mathbf{W}_u^2$, $\mathbf{W}_s^1$, and $\mathbf{W}_s^2$. Here we show the recurrence formula for such functions:

$$F_{k+1}(X) = f(X) - JF_k^{-1}(X), \qquad k \geqslant 0 \tag{9}$$

where F_0 is defined by the right-hand side of Eqs. (3)–(8) and $F_k^{-1}(X)$ is an inverse function. Here the index k implies the iteration time. The domain of the function $F_{k+1}(X)$ is confined by the region of the inverse function $F_k^{-1}(X)$.

We consider the iteration of the segment $[1/2\alpha, 1/2)$ on $\mathbf{W}_u^1$. This branch is mapped onto the branch expressed by

$$F_1 = -(a + J/\alpha)\, X + a \qquad (0.5 \leqslant X < \alpha/2) \tag{10}$$

Using Eq. (9), the succeeding branches are sequentially determined.

We also obtain the next branches of $\mathbf{W}_u^2$ by using Eq. (9) and Eqs. (5)–(6):

$$Y = (a + J/\alpha)\, X + J(X^* + Y^*/\alpha) \quad \text{connected with Eq. (6)} \tag{11}$$

$$Y = (a + J/\alpha)\, X - J(X^* + Y^*/\alpha) \quad \text{connected with Eq. (5)} \tag{12}$$

It is difficult to find full expressions for all branches of the stable and unstable manifolds even if the map is defined by a linear function. So we check their structures by numerical calculations. The numerical results are shown in Figs. 1 and 2, where the Jacobian is fixed to $J = 0.4$. In Fig. 1, the folded piecewise linear graph shows the unstable manifold $\mathbf{W}_u^1$ from the origin. The stable manifold $\mathbf{W}_s^2$ from the period-2 hyperbolic points is the boundary between two basins painted in black or in white. In Fig. 2, a portion of the unstable manifold $\mathbf{W}_u^2$ from the period-2 hyperbolic points is illustrated. From these figures, we find the following results:

1. The unstable manifold $\mathbf{W}_u^1$ from the origin is limited by $\mathbf{W}_u^2$ from the period-2 hyperbolic points.

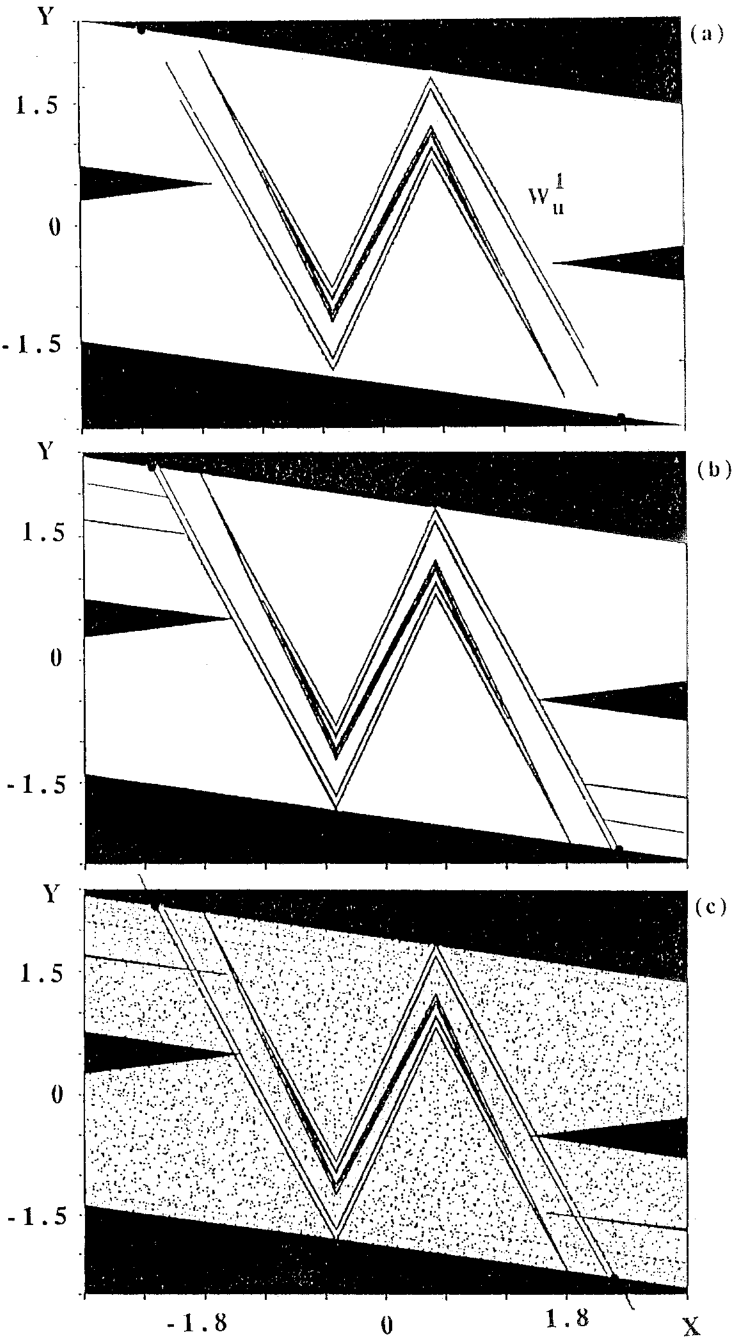

Fig. 1. Phase plane structure before and after the heteroclinic tangency in the piecewise linear map. (a) $a=2.4$: before the tangency; (b) $a=2.44924$: at tangency; (c) $a=2.48$: after tangency. $J=0.4$ in all cases. The piecewise linear graphs are the unstable manifold $\mathbf{W}_u^1$ starting from the origin. The stable manifold appears as the boundary of the black and white areas. The white area shows the basin of the attraction for the chaotic attractor and the black area is the basin for the attractor at infinity. In this case, the boundary is a simple curve. The period-2 hyperbolic points are indicated by filled circles.

2. The outermost branches of $\mathbf{W}_u^1$ accumulate to the stable manifold $\mathbf{W}_s^2$ at the situation of heteroclinic tangency. In this case, the heteroclinic tangency points are not on the manifolds.

3. The heteroclinic tangency between $\mathbf{W}_u^1$ and $\mathbf{W}_s^2$ and the homoclinic tangency between $\mathbf{W}_u^2$ and $\mathbf{W}_s^2$ occur at the same value of a.

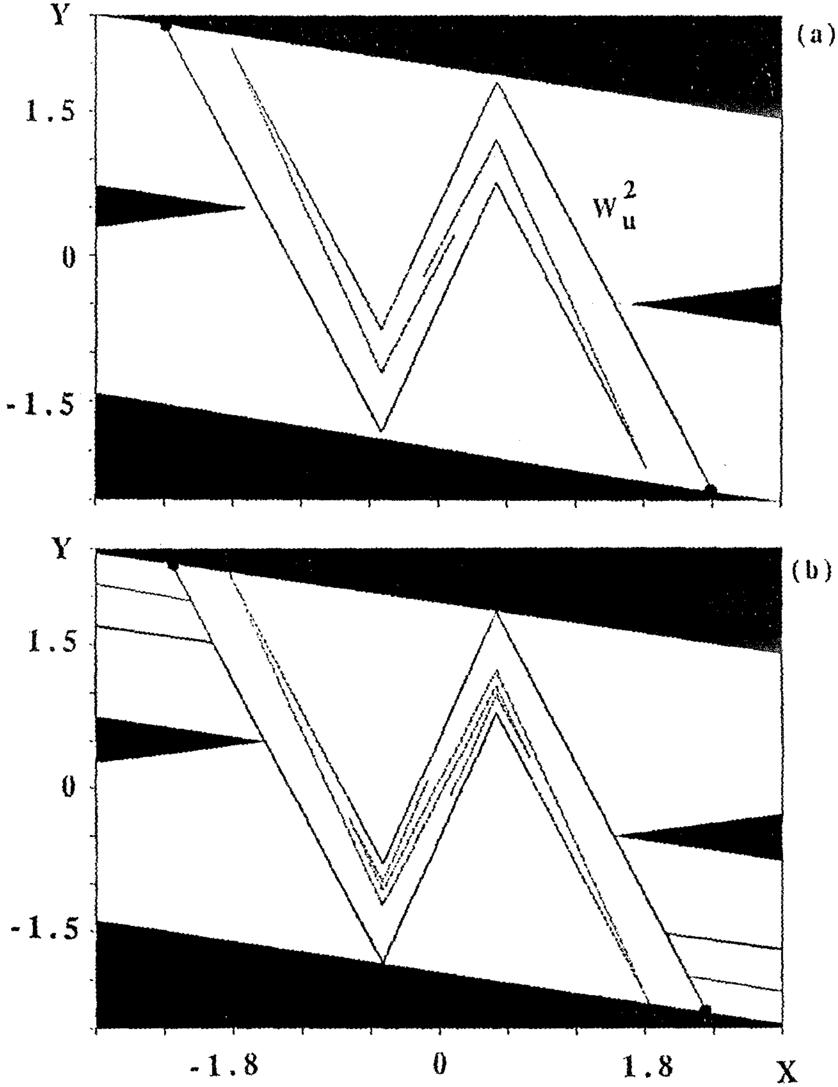

Fig. 2. The unstable manifold $\mathbf{W}_u^2$ from the period-2 hyperbolic points in the piecewise linear map. (a) $a = 2.4$ and (b) $a = 2.4492$. In both cases $J = 0.4$. The piecewise linear graphs are $\mathbf{W}_u^2$. See Fig. 1 for other explanations.

3. THE HETEROCLINIC TANGENCY FOR THE TWO-DIMENSIONAL CUBIC MAP

We consider another system with $f_a(Y) = aY - Y^3$ in Eq. (1). The origin is the hyperbolic fixed point without reflection when $a > J + 1$. The period-2 hyperbolic points without reflection $S_1(X^*, Y^*)$ and $S_2(-X^*, -Y^*)$ always exist:

$$Y^* = -X^* = (a + J + 1)^{1/2} \tag{13}$$

In refs. 7 and 8, the approximate expressions for $\mathbf{W}^1_{u,s}$ and $\mathbf{W}^2_{u,s}$ around the fixed points were given by using the functional equation method. We shall investigate the structure of the heteroclinic tangency by numerical calculations. For the case with $J = 0.4$, we have the critical values $a_h = 2.6419$ for the homoclinic tangency between $\mathbf{W}^1_u$ and $\mathbf{W}^1_s$ and $a_c = 2.7143$ for the heteroclinic tangency between $\mathbf{W}^1_u$ and $\mathbf{W}^2_s$. The homoclinic tangency between $\mathbf{W}^2_u$ and $\mathbf{W}^2_s$ occurs at $a_{h'} = 1.8863$. In Fig. 3, the area in white is the basin for the confined attractors and the area in black is the basin for the attractor at infinity. The boundary between the white and black areas is the stable manifold from the period-2 hyperbolic points.[8] Note that the stable manifold (basin boundary) accumulates toward itself in such cases; hence the basin boundary has the fractal structure. From Fig. 3, it is visible that the outermost branches of the unstable manifold accumulate to the fractal basin boundary (the stable manifold). After the heteroclinic tangency, the unstable manifold $\mathbf{W}^1_u$ suddenly spreads beyond $\mathbf{W}^2_s$. From the observation of Fig. 3, we have a conjecture that the heteroclinic tangency occurs between the accumulations of $\mathbf{W}^1_u$ and $\mathbf{W}^2_s$. In the next section, we study this conjecture in detail.

4. A THEOREM ON THE HETEROCLINIC TANGENCY

In Sections 2 and 3, we numerically found examples of maps in which the unstable manifold from a hyperbolic fixed point accumulates to the stable manifolds from period-2 points or to their accumulations.

In this section, we will justify our observation theoretically. Let us consider a one-parameter family of maps $T_a(X, Y)$:

$$T_a\colon\quad X_{n+1} = Y_n, \quad Y_{n+1} = f_a(Y_n) - JX_n \tag{1$'$}$$

where the Jacobian J is fixed at $0 < J \leqslant 1$ and the function $f_a(Y)$ satisfies the following conditions:

1. $f_a(Y)$ is an odd function and is unimodal at $Y > 0$.
2. The origin O is a hyperbolic fixed point without reflection.

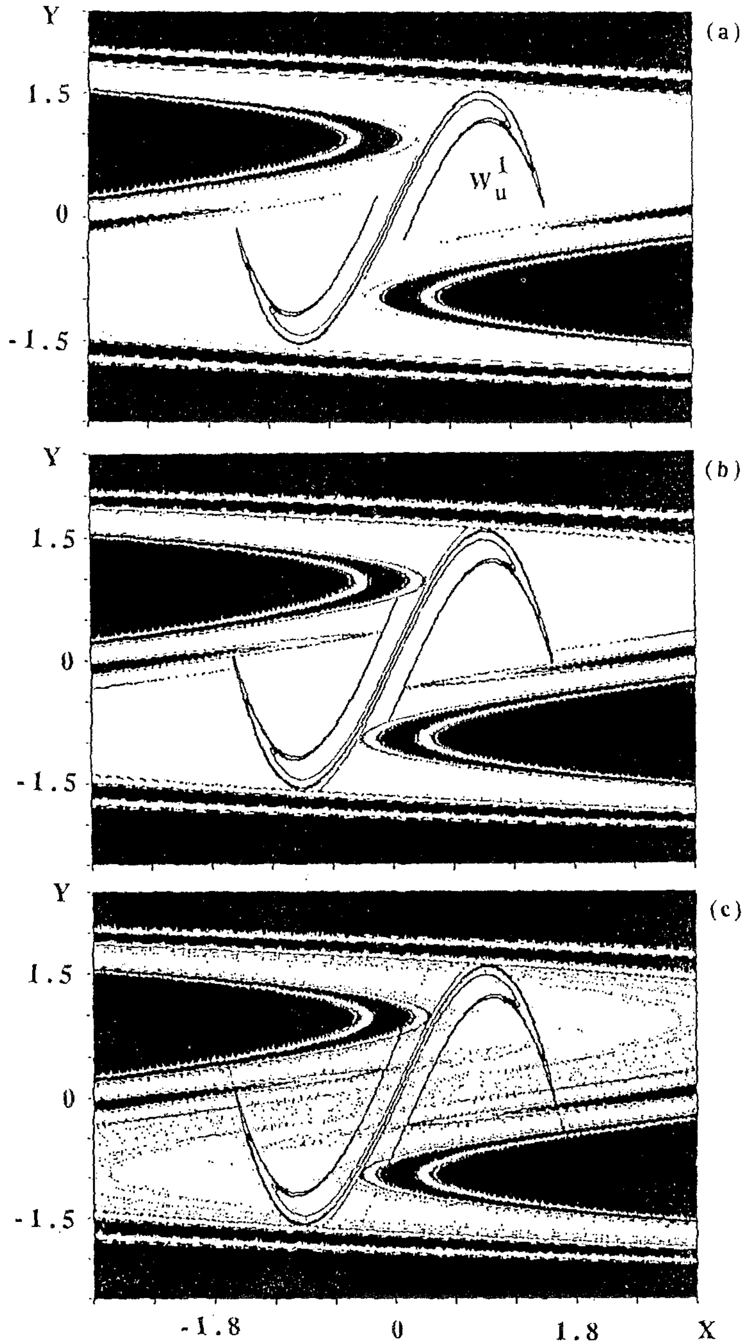

Fig. 3. Phase plane structure before and after the heteroclinic tangency in the cubic map. (a) $a = 2.68$: before tangency; (b) $a = 2.7143$: at tangency; (c) $a = 2.73$: after tangency. The Jacobian is fixed as $J = 0.4$. The stable manifolds from the period-2 periodic points appear as the boundary of the black and white areas. The white area shows the basin of the attraction for the periodic attractor and the black area is the basin for the attractor at infinity. In this case, the boundary is the fractal curve, and then it has the fuzzy structure.

3. There is a period-2 hyperbolic point without reflection. Two points S_1 and S_2 belonging to this periodic point are on the line $Y = -X$ (the Y coordinate of S_1 is positive).

4. In the one-dimensional limit with $J = 0$, the positions of period-2 hyperbolic points mentioned in condition 3 are the basin boundaries.

Hereafter the subscript a is omitted.

Next we introduce some notations. Let $\mathbf{W}_u^{1u}$ and $\mathbf{W}_u^{1d}$ be two branches of unstable manifolds outgoing from O, and $\mathbf{W}_s^{2l}$ be one of the stable manifolds ingoing to S_1, and $\mathbf{W}_s^{2r}$ the corresponding branch ingoing to S_2. Let $\mathbf{W}_u^1$ be one of $\mathbf{W}_u^{1u}$ and $\mathbf{W}_u^{1d}$ and $\mathbf{W}_s^2$ be one of $\mathbf{W}_s^{2l}$ and $\mathbf{W}_s^{2r}$. Let $[A]$ denote the closure of the set A. If $\mathbf{W}_u$ and $\mathbf{W}_s$ are tangent to each other at some point but never intersect at any point, we say that these manifolds are in the *first direct* heteroclinic tangency. We say that these manifolds are in the *first asymptotic* heteroclinic tangency if $\mathbf{W}_u$ and $\mathbf{W}_s$ have no common point, whereas $[\mathbf{W}_u]$ and $[\mathbf{W}_s]$ have common points.

Then our main result is stated as follows:

Theorem. Under the assumptions and notations stated above, the first direct heteroclinic tangency between $\mathbf{W}_u^1$ and $\mathbf{W}_s^2$ is impossible to occur in maps (1′).

Remark. Generically, intersections may be transverse (see Fig. 4a). However, we cannot at present exclude the case shown in Fig. 4b. The intersection stated in the theorem includes both cases as shown in Figs. 4a and 4b.

Proof. Suppose that $\mathbf{W}_u^{1u}$ and $\mathbf{W}_s^{2l}$ are tangent to each other at a point P, hence they are tangent to each other at the sequence of points $\{T^{2n}P\}(n = \pm 1, \pm 2,...)$. Let us assume that $\mathbf{W}_u^{1u}$ and $\mathbf{W}_s^{2l}$ do not intersect at any point, and derive a contradiction.

Let P' on $\mathbf{W}_u^{1d}$ be a point obtained from P by the coordinate change $X \to -X$ and $Y \to -Y$. Then, $\mathbf{W}_u^{1d}$ and $\mathbf{W}_s^{2r}$ are tangent to each other at the sequence of points $\{T^{2n}P'\}(n = \pm 1, \pm 2,...)$ due to the symmetry of the map (1′) coming from the oddness of f.

We make several remarks:

1. The map T is orientation preserving.[3]

2. The natural orientations of stable and unstable manifolds coincide at their common points. This can be easily shown with the aid of the continuity of the map. Thus the tangency shown in Fig. 5a is impossible to occur.

3. The map has the symmetry mentioned above. The point TP is on $\mathbf{W}_{\mathbf{u}}^{1u}$ and $\mathbf{W}_{\mathbf{s}}^{2r}$, and TP' is on $\mathbf{W}_{\mathbf{u}}^{1d}$ and $\mathbf{W}_{\mathbf{s}}^{2l}$. The point TP (resp. TP') is between P' (resp. P) and T^2P' (resp. T^2P) on both manifolds. Similarly, T^3P (resp. T^3P') is between T^2P' (resp. T^2P) and T^4P' (resp. T^4P) on both manifolds.

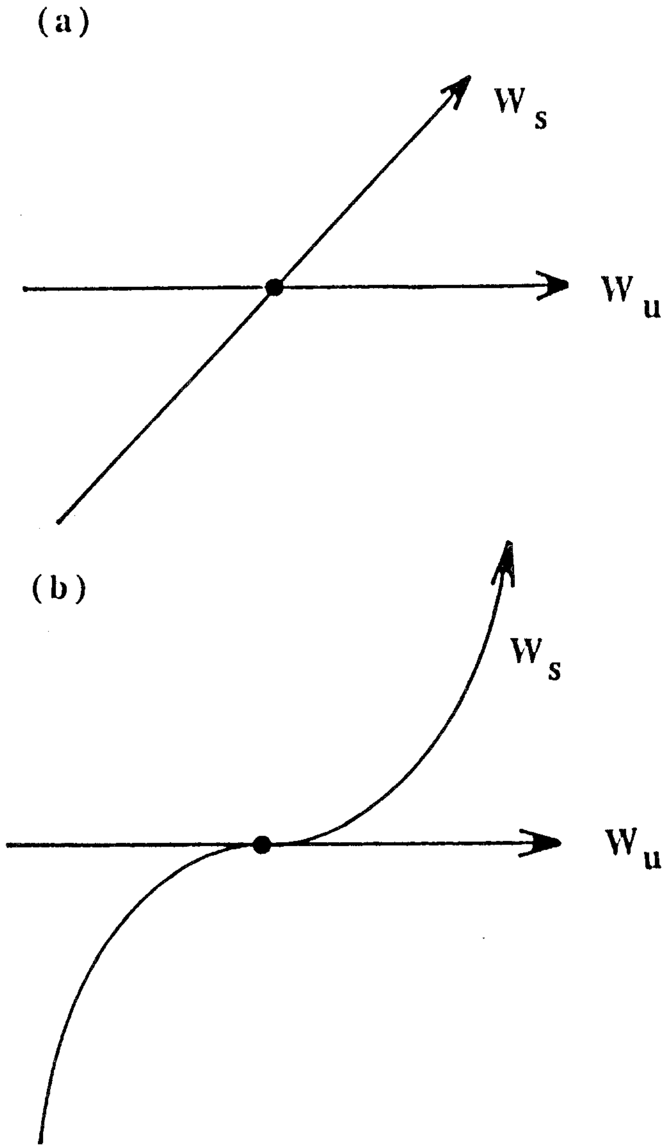

Fig. 4. Possible heteroclinic intersections. The intersection point is indicated by a filled circle.

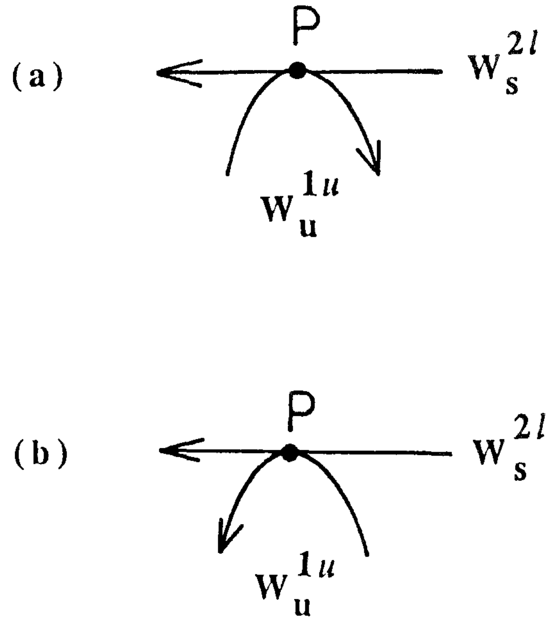

Fig. 5. Possible heteroclinic tangencies. (a) The orientations of two manifolds at P are opposite. (b) The orientations of two manifolds at P are the same.

4. The points $\{T^{2n}P\}(n = \pm 1, \pm 2,...)$ are on $\mathbf{W}_\mathbf{u}^{1u}$ and $\mathbf{W}_\mathbf{s}^{2l}$.

5. The stable manifold $\mathbf{W}_\mathbf{s}^{2l}$ is mapped onto $\mathbf{W}_\mathbf{s}^{2r}$ by T and vice versa.

6. The arc OP (resp. OP') of $\mathbf{W}_\mathbf{u}^{1u}$ (resp. $\mathbf{W}_\mathbf{u}^{1d}$) is a Jordan arc. The same is true with the arc PS_1 of $\mathbf{W}_\mathbf{s}^{2l}$ and the arc $P'S_2$ of $\mathbf{W}_\mathbf{s}^{2r}$.

We will show that $\mathbf{W}_\mathbf{u}^{1u}$ and $\mathbf{W}_\mathbf{s}^{2l}$ intersect. By remark 2, we only have to consider the case of Fig. 5b. From remarks 1–6, the situation of tangency is illustrated in Fig. 6. Let us denote by Γ the Jordan curve[11,12] formed by the arc PT^2P of $\mathbf{W}_\mathbf{u}^{1u}$ and the arc T^2PP of $\mathbf{W}_\mathbf{s}^{2l}$. Let us introduce the orientation on Γ as that of $\mathbf{W}_\mathbf{s}^{2l}$. The arrow **a** is on the right of Γ, whereas the arrow **b** is on the left. The points T^3P', T^4P, and S_1 are on the right of Γ. Thus the arc connecting the arrow **b** and T^3P' must intersect Γ because the arrow **b** and the point T^3P' are at the opposite sides of Γ. This arc (of $\mathbf{W}_\mathbf{u}^{1u}$) must intersect the arc PT^2P of $\mathbf{W}_\mathbf{s}^{2l}$ since two unstable manifolds cannot have a common point. Thus a contradiction is derived. This completes the proof. ∎

Here we discuss the meaning of the theorem. Suppose that there exist an unstable manifold $\mathbf{W}_\mathbf{u}^1$ and a stable manifold $\mathbf{W}_\mathbf{s}^2$ and that they are

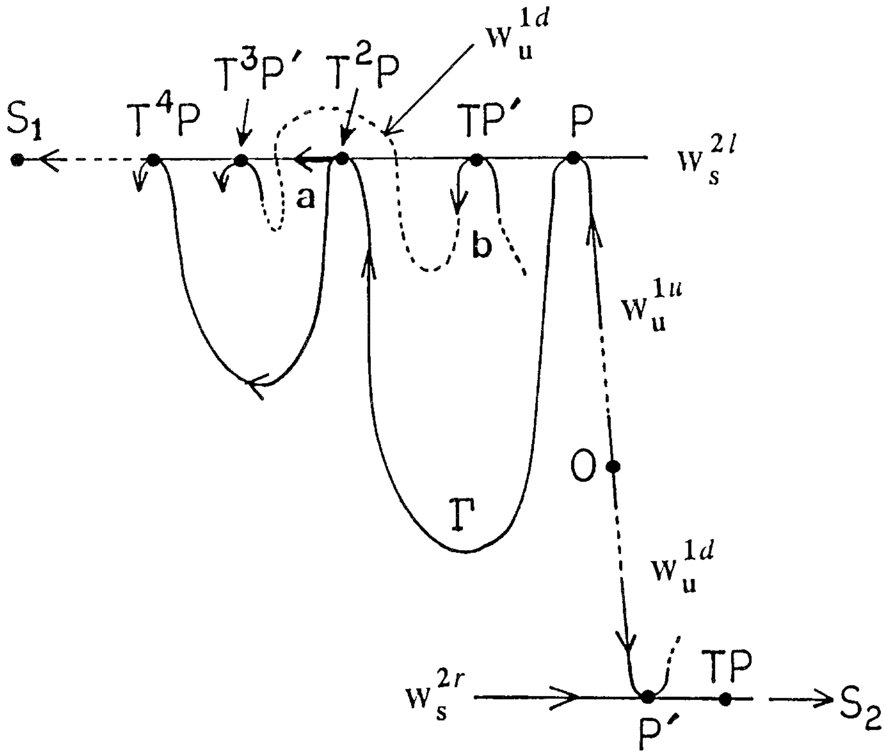

Fig. 6. Schematic illustrations of the unstable manifolds $\mathbf{W}_u^{1u}$ and $\mathbf{W}_u^{1d}$ and of the stable manifolds $\mathbf{W}_s^{2l}$ and $\mathbf{W}_s^{2r}$. The origin is denoted by O, and the period-2 hyperbolic points are shown by S_1 and S_2. See Section 4.

detached for $a < a_0$ and are heteroclinically intersecting for $a > a_0$. What happens at $a = a_0$? The usual belief is that the first direct heteroclinic tangency occurs. Our result is, contrary to the usual belief, that $\mathbf{W}_u^1$ and $\mathbf{W}_s^2$ cannot be in the first direct heteroclinic tangency and they are in the first asymptotic heteroclinic tangency. We discuss the asymptotic tangency in detail.

We define $\partial\mathbf{W}_u^1$ and $\partial\mathbf{W}_s^2$ as $[\mathbf{W}_u^1] - \mathbf{W}_u^1$ and $[\mathbf{W}_s^2] - \mathbf{W}_s^2$. To make clear these sets, we draw a schematic illustration in Fig. 7. The solid lines are the manifolds, for example, $\mathbf{W}_u$ and $\mathbf{W}_s$. The dashed lines indicate $\partial\mathbf{W}_u^1$ and $\partial\mathbf{W}_s^2$. It is to be noted that $\mathbf{W}_u^1$ and $\mathbf{W}_s^2$ may accumulate to themselves. In our definition, $\partial\mathbf{W}_u^1$ and $\partial\mathbf{W}_s^2$ do not contain the manifolds themselves.

Now the heteroclinic tangency between $\mathbf{W}_u$ and $\mathbf{W}_s$ is classified into four cases:

1. $\mathbf{W}_u$ and $\mathbf{W}_s$ touch at some points.

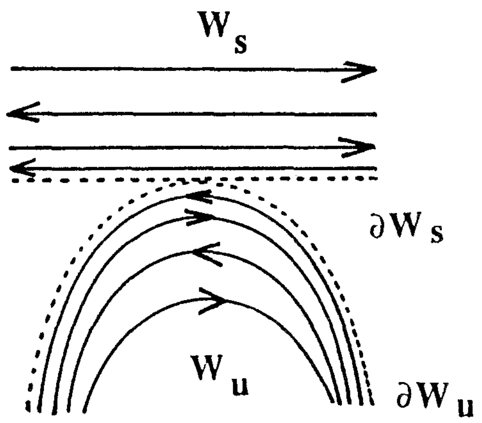

Fig. 7. Schematic illustration of $\mathbf{W}_u$ and $\mathbf{W}_s$ (shown by solid curves) and $\partial\mathbf{W}_u$ and $\partial\mathbf{W}_s$ (shown by the dashed lines).

2. $\partial\mathbf{W}_u$ and $\mathbf{W}_s$ touch at some points.
3. $\mathbf{W}_u$ and $\partial\mathbf{W}_s$ touch at some points.
4. $\partial\mathbf{W}_u$ and $\partial\mathbf{W}_s$ touch at some points.

Our theorem excludes case 1 for the map of Eq. (1′). In the following, we discuss the second and fourth cases illustrated in Figs. 1 and 3.

In Figs. 1 and 3, we observe the touching points of heteroclinic tangency, but do not find interesting points between the stable and unstable manifolds. Are the numerical results shown in these figures consistent with the theorem? The answer is that the asymptotic tangency occurs in these cases. The second case occurs in the piecewise linear map in Section 2, and the fourth case occurs in the cubic map in Section 3. What kind of heteroclinic tangency occurs depends on the structure of the invariant curves. In fact, in the cubic map, the homoclinic tangency between $\mathbf{W}_u^2$ and $\mathbf{W}_s^2$ occurs before the heteroclinic tangency, and the stable manifold $\mathbf{W}_s^2$ has a fractal structure after the homoclinic tangency. In this case, $\partial\mathbf{W}_u^1$ and $\partial\mathbf{W}_s^2$ touch. On the other hand, in the piecewise linear map, the structure of $\mathbf{W}_s^2$ is not fractal. In this case, $\partial\mathbf{W}_u^1$ and $\mathbf{W}_s^2$ touch.

Over the critical value of a, there exist infinitely many heteroclinic intersections and the heteroclinic points are on the manifolds.

5. CONCLUDING REMARK

Finally we discuss the physical importance of the heteroclinic tangency. Newhouse[13] proved that there is much more complicated dynamical behavior associated with a homoclinic tangency. We also expect that the complicated structure appears due to the heteroclinic tangency. After the heteroclinic tangency, the unstable manifold spreads beyond the stable manifold. For the case in Section 3, the heteroclinic tangency does not give rise to a serious effect on the orbit structure, because the stable attractors exist. If the chaotic attractor exists before the heteroclinic tangency occurs, we can expect the burst of the chaotic attractor. The heteroclinic tangency for the case in Section 3 gives rise to a change of the structure of basin boundaries.

After the heteroclinic tangency, a certain portion of $\mathbf{W}_u^1$ is on the other side of the stable manifold $\mathbf{W}_s^2$ (the basin boundary separating two basins). Consider the preimage of such a portion by T^{-n}. As n tends to infinity, the preimage converges to the origin $(0, 0)$. Before the heteroclinic tangency, the region close to the origin is the basin for confined attractors. After the tangency, there appear infinitely many areas in the vicinity of the origin which are the basin for the attractor at infinity. This implies that the basin for the attractor at infinity penetrates into the basin for the confined attractor. The basin boundary $\mathbf{W}_s^2$ has already been turned into a fractal curve by the homoclinic tangency at $a = 1.8863$. This fractal basin boundary changes into a more complicated fractal boundary (fractal–fractal metamorphosis[4,5]) by the heteroclinic tangency. A change of the fractal dimension of the basin boundary due to the heteroclinic tangency is found.[14]

ACKNOWLEDGMENTS

We would like to thank Prof. C. Grebogi for sending many papers. One of the authors (Y. Y.) thanks Dr. M. Kawano for helpful discussions. This work is partly supported by a Grand-in-Aid of Teikyo University of Technology.

REFERENCES

1. C. Grebogi, E. Ott, and J. A. Yorke, *Science* **238**:585 (1987).
2. C. Mira, *Chaotic Dynamics* (World *Scientific*, Singapore, 1987).
3. J. Guckenheimer and P. J. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, 1983).
4. C. Grebogi, E. Ott, and J. A. Yorke, *Physica* **24D**:243 (1987).
5. S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, *Physica* **17D**:125 (1985).
6. Y. Yamaguchi and N. Mishima, *Phys. Lett. A* **104**:179 (1984).
7. Y. Yamaguchi, *Phys. Lett. A* **133**:201 (1988).

8. Y. Yamaguchi, *Phys. Lett. A* **135**:259 (1989).
9. R. Lozi, *J. Phys.* (Paris) **39**:C5-9 (1978).
10. M. Henon, *Commun. Math. Phys.* **50**:69 (1976).
11. K. Kodaira, *Introduction to Complex Analysis* (Cambridge University Press, 1984).
12. K. Tanikawa and Y. Yamaguchi, *J. Math. Phys.* **28**:921 (1987).
13. S. E. Newhouse, *Topology* **13**:9 (1974); *Publ. Math. IHES* **50**:101 (1979).
14. Y. Yamaguchi and K. Tanikawa, *Phys. Lett. A* **142**:95 (1989).